MATH 2243: **Linear Algebra** & **Differential Equations**

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9.1: Stability and the Phase Plane: Review

Autonomous System

Two-dimensional first order SoDEs of the form: $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$.

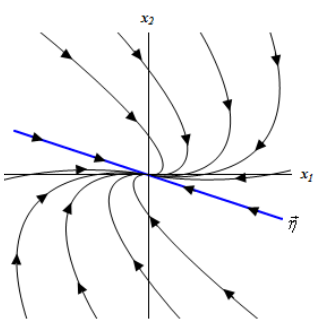
Since these derivatives do not explicitly depend upon t (time),
it is referred to as an **autonomous system**.

Solution curves to such a system in the phase plane is called a **trajectory**.

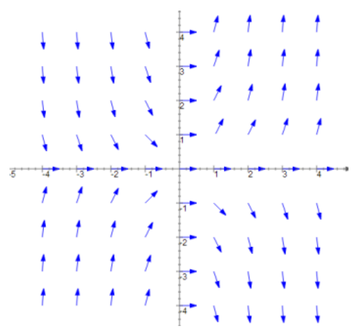
A **critical point** (x_*, y_*) is a point that does not move with time…
$F(x_*, y_*) = G(x_*, y_*) = 0$.

As a result, $x(t) = x_*$ and $y(t) = y_*$ is a solution to the system
and is called an **equilibrium solution**.

Phase Portraits



Slope/Direction Fields

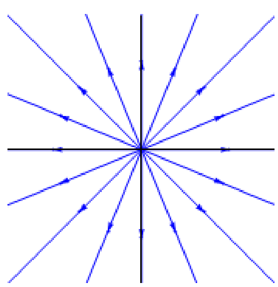
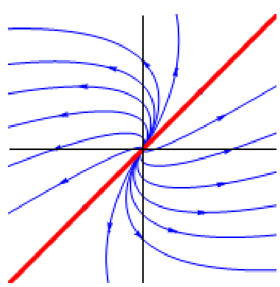


Determining Critical Points:
Set: $F(x,y) = 0$ and $G(x,y) = 0$ and solve for x and y.

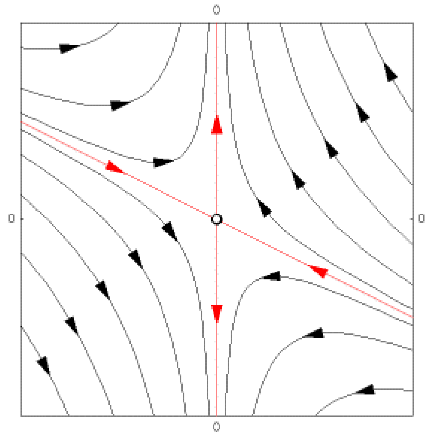
Node: Every trajectory approaches (recedes) from $(x_*,\ y_*)$ as $t \to \infty$, AND
every trajectory is tangent at $(x_*,\ y_*)$ to some straight line through $(x_*,\ y_*)$.

Proper Node: Trajectories approach or recede in all directions.

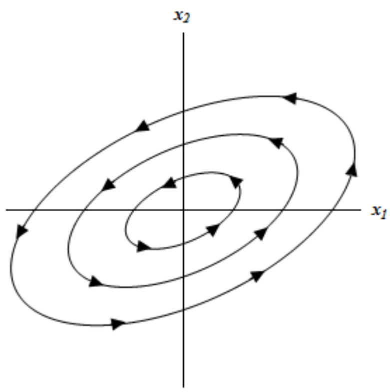
Improper Node: All trajectories approach or recede in just two directions.

Sink: All trajectories approach the critical point.
Source: All trajectories recede from the critical point.
Saddle Point: Two trajectories approach the critical point, but all others are unbounded as $t \to \infty$.



Stable Critical Point: For each $\varepsilon > 0$, there exists $\delta > 0$ such that
$\left|\vec{x}_0 - (x_*,\ y_*)\right| < \delta$ results in $\left|\vec{x}(t) - (x_*,\ y_*)\right| < \varepsilon$ for all $t > 0$.
Unstable: Simply means the critical point is not stable.
Center: Is when $(x_*,\ y_*)$ is stable and surrounded by simple closed trajectories.



Asymptotically Stable: $(x_*,\ y_*)$ is stable and every trajectory that begins sufficiently close to $(x_*,\ y_*)$, also approaches $(x_*,\ y_*)$ as $t \to \infty$.
Stable Spiral Point or **Spiral Sink**: Asymptotically stable critical point around which trajectories spiral as they approach.
Unstable Spiral Point or **Spiral Source**: Asymptotically stable critical point around which trajectories spiral as they recede.
Closed Trajectory: Simple closed solution curve representing a periodic solution (like the elliptical trajectories above).

Problem: #5 Find the critical point or points of $\frac{dx}{dt} = 1 - y^2, \qquad \frac{dy}{dt} = x + 2y.$

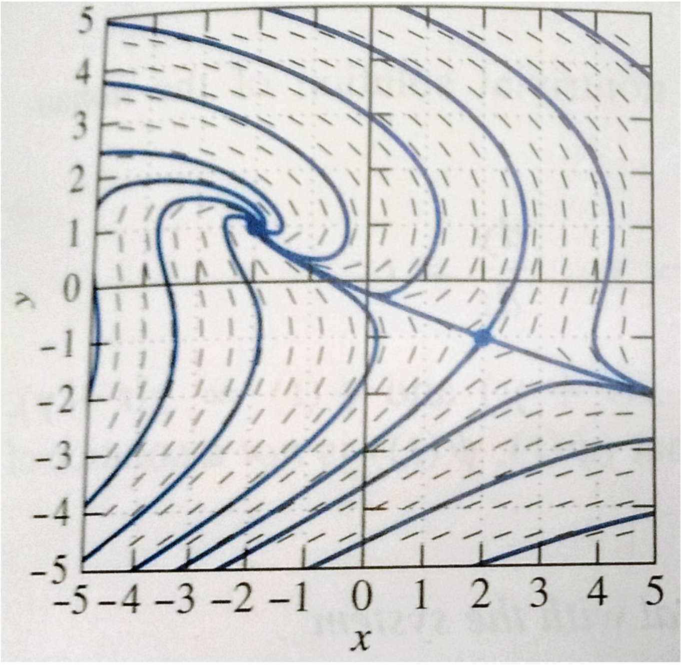
$0 = 1 - y^2 \qquad\qquad 0 = x + 2y$

Gives $y = 1$ or $y = -1$.

$0 = x + 2(1)$ or $0 = x + 2(-1)$ gives $x = -2$ or $x = 2,$ respectively.

So, $(-2, 1)$ and $(2, -1)$ are our critical points.

See the graph below…



Problem: #12 Find the equilibrium solution $x(t) \equiv x_0$ of: $x'' + (x^2 - 1)x' + x = 0.$

Construct a phase portrait and direction field for the equivalent first order system:

$x' = y, \qquad y' = -f(x, y).$

Ascertain whether the critical point $(x_0, 0)$ looks like a center, saddle point, or spiral point.

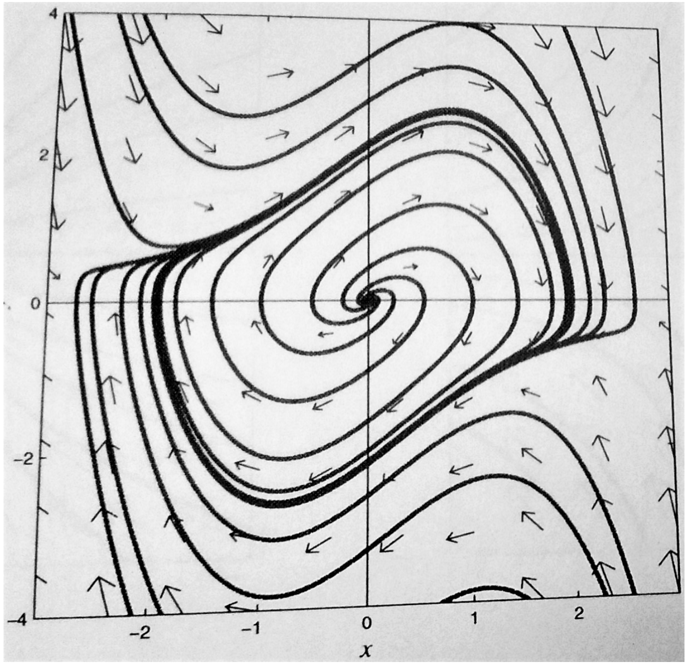
Set $x' = x'' = 0$, solve for x.

$x = 0$ and thus the single equilibrium solution $x(t) \equiv 0.$

Phase plane portrait:

$x' = y, \qquad y' = -(x^2 - 1)y - x$ is shown below.

We observed that the critical point $(0, 0)$ in the phase plane looks like a spiral source, with the solution curves emanating from this source spiraling outward toward a closed curve trajectory.



Problem: **#20** Solve the system to determine whether the critical point $(0, 0)$ is stable, asymptotically stable, or unstable.

Construct a phase portrait and direction field for the given system.

Ascertain the stability or instability of each critical point, and identify it visually as a node, a saddle point, a center, or a spiral point.

$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -5x - 4y$

Substitution of $y' = x''$ from the first equation into the second one gives $x'' = -5x - y = -5x - 4x'$,

so $x'' + 4x' + 5x = 0$.

The characteristic roots of this equation are $r = -2 \pm i$, so we get the general solution…

$x(t) = e^{-2t}(A\cos t + B\sin t)$ and $y(t) =$?

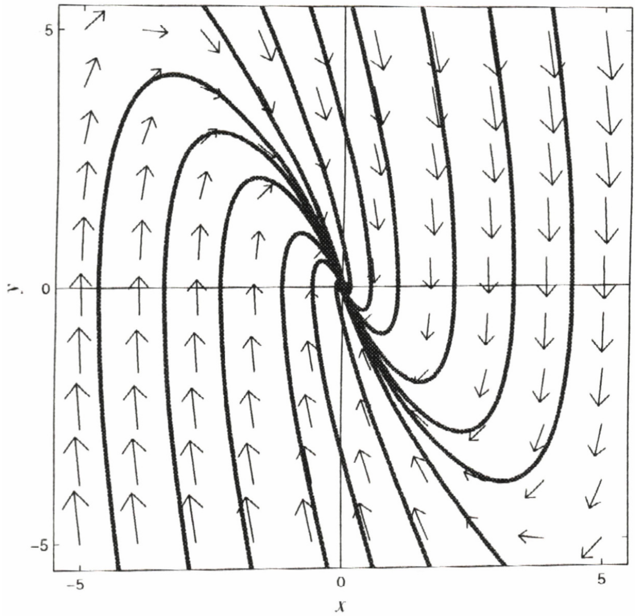
$y(t) = e^{-2t}[(-2A + B)\cos t - (A + 2B)\sin t]$

(the latter because $y = x'$).

Origin stable?

Clearly $x(t),\ y(t) \to 0$ as $t \to +\infty$, so the origin is stable.

Below you see the origin is an asymptotically stable spiral point with trajectories approaching $(0, 0)$.



Problem: **#26** Given the system: $\frac{dx}{dt} = y^3 e^{x+y}$, $\frac{dy}{dt} = -x^3 e^{x+y}$.

Solve the equation: $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$ to find the trajectories of the given system.

Construct a phase portrait and direction field for the system.

Identify visually the apparent character and stability of the critical point $(0,0)$ of the system.

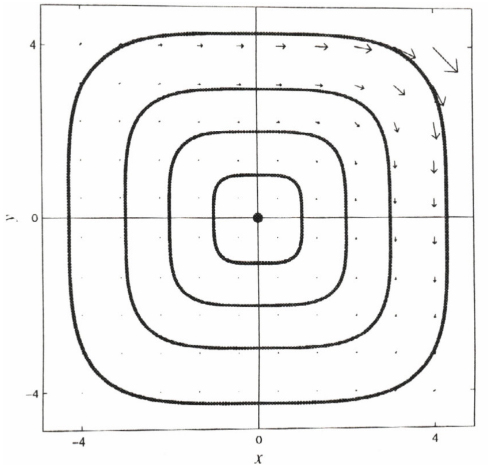
$\frac{dy}{dx} = -\frac{x^3}{y^3}$

Separates to: $y^3 dy = x^3 dx$

$\int y^3 dy = \int x^3 dx \qquad \frac{1}{4}y^4 = \frac{1}{4}x^4 + C'$.

So $x^4 + y^4 = C$.

Thus the trajectories consist of the origin $(0,0)$ and the ovals of the form $x^4 + y^4 = C$, as illustrated below…



Closed periodic trajectories.